



Raffles Mathematical Olympiad 2022

Round 1 (Solutions)

Date: 29 March 2022

Duration: 1 hour

This paper consists of 20 questions.

*For practice purpose, the multiple choice options are removed.

The marks allocation is as follows:

| Question Number | Correct | Unanswered | Incorrect |
|-----------------|---------|------------|-----------|
| 1 to 10 | 4 marks | 1 mark | 0 mark |
| 11 to 20 | 6 marks | 1 mark | 0 mark |

1.

$$\begin{aligned} & \left(1 + \frac{1}{3}\right) \times \left(1 + \frac{1}{4}\right) \times \left(1 + \frac{1}{5}\right) \times \dots \times \left(1 + \frac{1}{n}\right) \\ &= \frac{4}{3} \times \frac{5}{4} \times \frac{6}{5} \times \dots \times \frac{n}{n-1} \times \frac{n+1}{n} \\ &= \frac{n+1}{3} \end{aligned}$$

$$\frac{n+1}{3} > 2022$$

$$n+1 > 6066$$

$$\Rightarrow n > 6065$$

Smallest n is 6066, so sum of digits of $n = \boxed{18}$.

2.

$$\begin{aligned} \frac{23}{130} &= \frac{13}{130} + \frac{10}{130} = \frac{1}{10} + \frac{1}{13} \\ &= 0.1 + 0.076923076923076923\dots\dots \\ &= 0.1769230769230769230\dots\dots \end{aligned}$$

Since $2022 = 1 + 6 \times 336 + 5$, the 2022th digit is $\boxed{3}$.

3.

$$\begin{aligned} & \overline{123abc789} \\ &= 123 \times 1000000 + \overline{abc} \times 1000 + 789 \\ &= 123 \times (999999 + 1) + \overline{abc} \times (999 + 1) + 789 \\ &= (123 \times 999999 + \overline{abc} \times 999) + (123 + \overline{abc} + 789) \end{aligned}$$

For the number to be divisible by 999, $\overline{abc} = 999 - 789 - 123 = \overline{087}$,
so $a + b + c = \boxed{15}$.

4.

$$\begin{aligned} & \frac{2022}{674 + 674^2} + \frac{2022}{675 + 675^2} + \frac{2022}{676 + 676^2} + \dots + \frac{2022}{1010 + 1010^2} \\ &= 2022 \left(\frac{1}{674 \times 675} + \frac{1}{675 \times 676} + \frac{1}{676 \times 677} + \dots + \frac{1}{1010 \times 1011} \right) \\ &= 2022 \left(\frac{1}{674} - \frac{1}{675} + \frac{1}{675} - \frac{1}{676} + \frac{1}{676} - \frac{1}{677} + \dots + \frac{1}{1010} - \frac{1}{1011} \right) \\ &= 2022 \left(\frac{1}{674} - \frac{1}{1011} \right) = 3 - 2 = \boxed{1}. \end{aligned}$$

5.

$$\begin{aligned} & \frac{8088}{24} + \frac{8088}{40} + \frac{8088}{60} + \frac{8088}{84} + \frac{8088}{112} + \frac{8088}{144} + \frac{8088}{180} + \frac{8088}{220} + \frac{8088}{264} \\ &= 2022 \left[\frac{1}{6} + \left(\frac{1}{10} + \frac{1}{15} \right) + \left(\frac{1}{21} + \frac{1}{28} \right) + \left(\frac{1}{36} + \frac{1}{45} \right) + \left(\frac{1}{55} + \frac{1}{66} \right) \right] \\ &= 2022 \left[\frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} \right] \\ &= 2022 \left[\frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} \right] \\ &= \boxed{1011}. \end{aligned}$$

6.

Ratio of diameters = 1 : 2 : 3

Ratio of areas of circles = 1 : 4 : 9 = S : 4S : 9S

$$\text{Area of unshaded parts in cm}^2 = 9S - 4S - S + S + S = 6S = 6 \times \frac{22}{7} \times \left(\frac{7}{2} \right)^2 = \boxed{231}.$$

7.

$$\begin{aligned} N &= 2022 + \frac{2022}{1+2} + \frac{2022}{1+2+3} + \frac{2022}{1+2+3+4} + \dots + \frac{2022}{1+2+3+4+\dots+2021} \\ &= 2022 \left(1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+4+\dots+2021} \right) \\ &= 2022 \left[2 \left(\frac{1}{1} - \frac{1}{2} \right) + 2 \left(\frac{1}{2} - \frac{1}{3} \right) + 2 \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + 2 \left(\frac{1}{2021} - \frac{1}{2022} \right) \right] \\ &= 2022 \left[2 \left(1 - \frac{1}{2022} \right) \right] \\ &= 4044 - 2 \\ &= 4042 \end{aligned}$$

$$\text{Sum of digits} = 4 + 4 + 2 = \boxed{10}.$$

8.

$$\text{Sum} = 30 \times 2022 + \frac{6}{2}(5 + 7 + 9 + 11 + 13) = \boxed{60795}.$$

9.

Let x be the number of male contestants in 2015.

Let y be the number of female contestants in 2015.

In 2016,

$$\text{No. of male contestants} = \frac{6}{5}x$$

$$\text{No. of female contestants} = \frac{9}{5}y$$

$$\frac{6}{5}x + \frac{9}{5}y = \frac{33}{25}(x + y)$$

$$\frac{3}{5}x = \frac{12}{5}y$$

$$x = 4y$$

$$\text{Required fraction} = \frac{\frac{9}{5}y}{\frac{6}{5}x + \frac{9}{5}y} = \frac{\frac{9}{5}y}{\frac{6}{5}(4y) + \frac{9}{5}y} = \frac{9}{33} = \boxed{\frac{3}{11}}.$$

10.

Let Mary's speed be x km/h.

Let the perimeter of the track be y km.

$$\text{Suppose both of them ride anticlockwise: } (36 - x) \times \frac{80}{60} = \frac{y}{2}$$

$$\text{Suppose Esther rides anticlockwise and Mary rides clockwise: } (36 + x) \times \frac{10}{60} = \frac{y}{2}$$

$$(36 - x) \times \frac{80}{60} = (36 + x) \times \frac{10}{60}$$

$$288 - 8x = 36 + x$$

$$9x = 252$$

$$x = 28$$

$$\text{So, perimeter} = y = \frac{64}{3} = \boxed{21\frac{1}{3} \text{ km}}.$$

11.

Let the number of litres of water drank per person be m and the number of days the amount of water distributed can last be n .

$$mn = (m - 1)(n + 15) \dots\dots(1)$$

$$mn = (m - 1.5)(n + 30)\dots\dots(2)$$

From (1): $15m - n = 15$

From (2): $30m - 1.5n = 45$

Solving, $0.5n = 15$

$$\Rightarrow n = 30$$

Therefore, the water distributed can last **30 days**.

12.

Eric wears yellow clothes on odd days and Jasmine wears yellow when the date is a multiple of 3.

So both of them wear yellow clothes on 3rd, 9th, 15th, 21st and 27th day of each month, and the number of days when both wear clothes of different colours is given by

$$365 - 5 \times 12 = \mathbf{305 \text{ days}}$$

13.

| | | |
|---|---|---|
| X | | X |
| | X | |
| X | | X |

Among the first 9 integers, 5 odd integers and 4 even integers. Clearly to satisfy the requirement, we need to fill up the cells labelled "X" with odd integers and the rest with even integers.

$$\text{Number of ways} = (5 \times 4 \times 3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1) = 120 \times 24 = \mathbf{2880}$$

14.

Divide the 30 consecutive positive integers into 3 groups.

Group 1 : When divided by 3, remainder is 0

Group 2 : When divided by 3, remainder is 1

Group 3 : When divided by 3, remainder is 2

Each group has 10 numbers.

For the sum of the 3 numbers to be divisible by 3, the 3 numbers must be chosen in the following ways :

Case 1 : all 3 numbers from group 1, 2 or 3

$$\text{No of ways} = \binom{10}{3} \times 3$$

Case 2 : 1 number from each group

$$\text{No of ways} = \binom{10}{1} \times \binom{10}{1} \times \binom{10}{1}$$

$$\text{Therefore the number of ways} = \binom{10}{3} \times 3 + \binom{10}{1} \times \binom{10}{1} \times \binom{10}{1} = 360 + 1000 = \boxed{1360}.$$

15.

Let the original height of water be x cm.

Let the lengths of rulers B and C be b cm and c cm respectively.

After adding some water, the height of water be $2x$ cm.

From the given information,

$$20 - x : b - x : c - x = 1:2:4$$

$$20 - 2x : b - 2x : c - 2x = 1:3:7$$

$$\frac{20 - x}{c - x} = \frac{1}{4}$$

$$80 - 4x = c - x$$

$$c = 80 - 3x$$

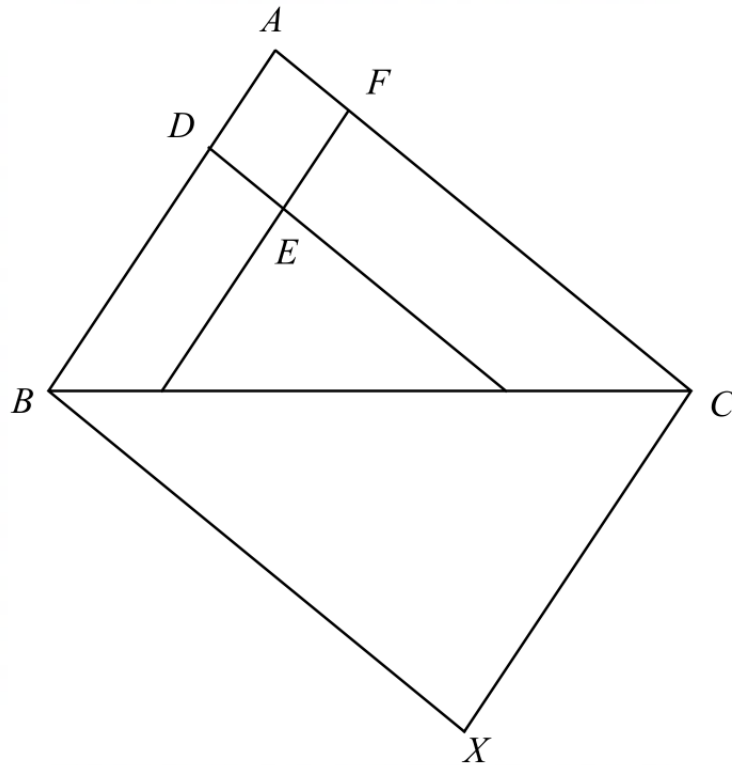
$$\frac{20 - 2x}{c - 2x} = \frac{1}{7}$$

$$140 - 14x = c - 2x$$

$$140 - 14x = 80 - 3x - 2x$$

$$x = \frac{20}{3} = \boxed{6\frac{2}{3}}.$$

16.



Let X be the point such that BX is parallel to AC and CX is parallel to AB . Then the parallelograms $ABXC$ and $ADEF$ have equal corresponding angles i.e. they are similar.

Hence area of triangle ABC

$$\begin{aligned}
 &= \frac{1}{2} \times \frac{AB}{AD} \times \frac{AC}{AF} \times (\text{area of parallelogram } ADEF) \\
 &= \frac{1}{2} \times \frac{26}{7} \times \frac{33}{6} \times 14 \\
 &= \boxed{143 \text{ cm}^2}.
 \end{aligned}$$

17.

$$\text{Let } S = \frac{1}{5} + \frac{1}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \frac{5}{5^5} + \frac{8}{5^6} + \dots \quad (1)$$

$$\text{Then } \frac{S}{5} = \frac{1}{5^2} + \frac{1}{5^3} + \frac{2}{5^4} + \frac{3}{5^5} + \frac{5}{5^6} + \frac{8}{5^7} + \dots \quad (2)$$

$$\begin{aligned}
 (1) - (2): \quad \frac{4}{5}S &= \frac{1}{5} + \left(\frac{1}{5^3} + \frac{1}{5^4} + \frac{2}{5^5} + \frac{3}{5^6} + \dots \right) \\
 &= \frac{1}{5} + \frac{1}{5^2} \left(\frac{1}{5} + \frac{1}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \dots \right)
 \end{aligned}$$

$$\frac{4}{5}S = \frac{1}{5} + \frac{1}{25}S$$

$$\text{Thus, } S = \boxed{\frac{5}{19}}.$$

18.

Let $\angle BCA = p^\circ$ and $\angle FCA = q^\circ$

$$p + q = 46$$

$\angle CAD = p^\circ$ (alternate \angle s)

$$\text{Since } AC = AD, \angle ACD = \frac{180^\circ - p^\circ}{2}$$

$$\text{Since } AC = CF, \angle CAF = \frac{180^\circ - q^\circ}{2} = \angle ECA \text{ (alt } \angle\text{s)}$$

$$\angle ECD = \frac{180^\circ - p^\circ}{2} + \frac{180^\circ - q^\circ}{2} = \frac{360^\circ - (p^\circ + q^\circ)}{2} = \frac{360^\circ - 46^\circ}{2} = \boxed{157^\circ}.$$

19.

For a child with label n , we consider the factors of n .

For each odd factor, 3 candies are added and for each even factor, 1 candy is taken away. So the child has net addition of candies if the number of odd factors exceeds 3 times the number of even factors of n and vice versa.

Write $n = 2^r s$, s : odd.

So every odd factor of n corresponds to r even factors of n .

Hence the child with label n gets extra candies if and only if $r < 3$ ie. n is not divisible by 8. Since there are 125 multiples of 8 among the first 1000 positive integers, the answer is $1000 - 125 = \boxed{875}$.

20.

Let the number of \$1.4, \$1.8, \$2.4 and \$3 be a, b, c and d respectively.

To solve $1.4a + 1.8b + 2.4c + 3d = 12$

i.e. $7a + 9b + 12c + 15d = 60$

where a, b, c, d are non negative integers.

Since all the coefficients are divisible by 3 except 7, a must be divisible by 3.
So possible values for a are 0, 3 or 6.

1°: $a = 6$. Then equation reduces to $3b + 4c + 5d = 6$

There is only 1 solution: $b = 2, c = d = 0$.

2°: $a = 3$. Then equation reduces to $3b + 4c + 5d = 13$

The possible solutions are:

$$d = 2, b = 1, c = 0$$

$$d = 1, c = 2, b = 0$$

$$d = 0, b = 3, c = 1$$

3°: $a = 0$. Then equation reduces to $3b + 4c + 5d = 20$

The possible solutions are:

$$d = 4, b = 0, c = 0$$

$$d = 3 \text{ (no solutions)}$$

$$d = 2, c = 1, b = 2$$

$$d = 1, b = 1, c = 3$$

$$d = 1, b = 5, c = 0$$

$$d = 0, b = 0, c = 5$$

$$d = 0, b = 4, c = 2$$

In total, there are **10 ways**.