2026 Spring Cup

Mathematical Olympiad

PRELIMINARY ROUND

Date: 31 January 2026

Time Given: 1 hour 30 minutes

Level: Secondary Junior

Name: ________________

Parent's Phone Number: ________________

Instructions to Candidates

1. Do not open the booklet until you are told to do so.
2. Answer ALL 15 questions.
3. Write your answers in the answer sheet provided.
4. No steps are needed to justify your answers.
5. Questions 1-3 are worth 8 marks each.
6. Questions 4-14 are worth 10 marks each.
7. Questions 15 is worth 16 marks.
8. No marks will be deducted for wrong answers.
9. No marks will be given for unanswered questions.
10. No calculators or mathematical instruments are allowed.

I. ***Short Answer Questions(1) (8 marks each, 24 marks in Total)***

1. If $a=3^{110}$, $b=4^{88}$, $c=5^{66}$, find the letter with the greatest value.

【Answer】 b

【Solution】

HCF(110,88,66) = 22

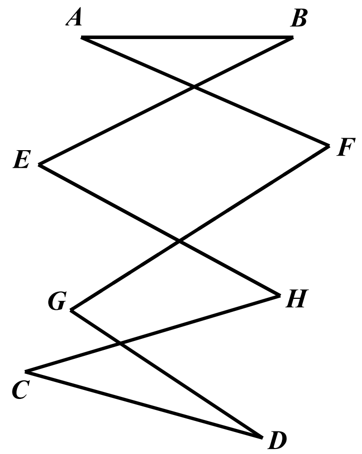
$a=3^{110}=(3^5)^{22}=(243)^{22}$,

$b=4^{88}=(4^4)^{22}=(256)^{22}$,

$c=5^{66}=(5^3)^{22}=(125)^{22}$

Hence $b>a>c$

2. As shown in the figure, if $\angle E = 55°$, $\angle F = 56°$, $\angle G = 66°$, and $\angle A + \angle B = \angle C + \angle D$, find the value of $\angle H$.



【Answer】45°

【Solution】

Suppose $\angle A+\angle B=\angle C+\angle D=180°-\theta$,

Then $\theta+\angle E+\angle F=\theta+\angle G+\angle H$,

$$\begin{aligned}\angle H &= \angle E+\angle F-\angle G\\ &=55°+56°-66°\\ &=45°\end{aligned}$$

3. If real number a satisfy that $|2025-a|+\sqrt{a-2026}=a$, find the value of $a - 2025^2$.

【Answer】2026

【Solution】

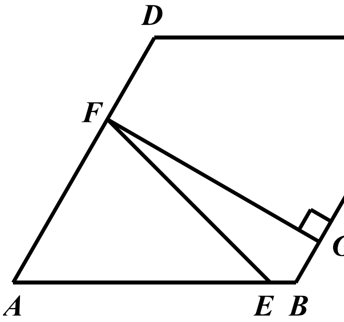
$\sqrt{a-2026}$ exists $\Rightarrow a-2026\ge 0\Rightarrow a\ge 2026$

$|2025-a|=a-2025$

$$\begin{aligned}a-2025+\sqrt{a-2026}&=a\\ \sqrt{a-2026}&=2025\\ a-2026&=2025^2\\ a-2025^2&=2026\end{aligned}$$

II. *Short Answer Questions(2) (10 marks each, 110 marks in Total)*

4. As shown in the figure, in rhombus $ABCD$, $AB = 6$, $DF = 2$, $\angle DAB = 60°$, $\angle EFG = 15°$, $FG \perp BC$, find the length of AE.



【Answer】 $2+2\sqrt{3}$

【Solution】

From point F, draw a line perpendicular to AB at point H.

$AF = AD - FD = 4$, and since $\angle DAB = 60°$,

we have $\angle AFH = 30°$, $AH = 2$, $FH = 2\sqrt{3}$.

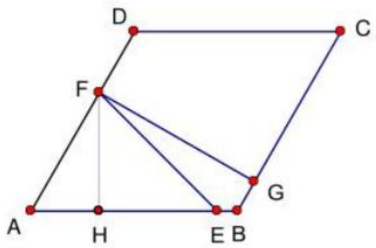
Also $\angle EFG = 15°$, hence

$\angle EFH = \angle AFG - \angle AFH - \angle EFG = 90° - 30° - 15° = 45°$,

Hence $\triangle FHE$ is an isosceles right triangle, and

$HE = FH = 2\sqrt{3}$,

Thus, $AE = AH + HE = 2 + 2\sqrt{3}$.



5. If rational number $a_1, a_2, a_3, \cdots, a_{2026}$ satisfy that $a_1 + a_2 + \cdots + a_{2026} = 0$, find how many different possible values of $\frac{a_1}{|a_1|} + \frac{a_2}{|a_2|} + \frac{a_3}{|a_3|} + \cdots + \frac{a_{2026}}{|a_{2026}|}$.

【Answer】2025

【Solution】

$\frac{a_n}{|a_n|} = \pm 1$

Suppose there are m negative numbers in $a_1, a_2, a_3, \cdots, a_{2026}$,

$a_1 + a_2 + \cdots + a_{2026} = 0 \Rightarrow 1 \le m \le 2025$

When $m = 1$,

$\frac{a_1}{|a_1|} + \frac{a_2}{|a_2|} + \frac{a_3}{|a_3|} + \cdots + \frac{a_{2026}}{|a_{2026}|} = (-1) \times 1 + 1 \times 2025 = 2024$

When the value of m increases by 1,

the resulting value $\frac{a_1}{|a_1|} + \frac{a_2}{|a_2|} + \frac{a_3}{|a_3|} + \cdots + \frac{a_{2026}}{|a_{2026}|}$ decreases by 2.

Hence for 2025 distinct values of m, we have 2025 different resulting values.

6. If $0 < a < 1$, and satisfy that $\left\lfloor a+\frac{1}{50}\right\rfloor+\left\lfloor a+\frac{2}{50}\right\rfloor+\ldots+\left\lfloor a+\frac{49}{50}\right\rfloor=29$, ($\lfloor x\rfloor$ means the greatest integer less than or equal to x), find the value of $\lfloor 10a\rfloor$.

【Answer】5

【Solution】

$0<a<1\Rightarrow\left\lfloor a+\frac{n}{50}\right\rfloor$ is either 0 or 1, for integer $n = 1, 2, 3, \ldots, 49$.

Also, $\left\lfloor a+\frac{n}{50}\right\rfloor\le\left\lfloor a+\frac{n+1}{50}\right\rfloor$.

Then, given $\left\lfloor a+\frac{1}{50}\right\rfloor+\left\lfloor a+\frac{2}{50}\right\rfloor+\ldots+\left\lfloor a+\frac{49}{50}\right\rfloor=29$

$\Rightarrow\left\lfloor a+\frac{n}{50}\right\rfloor=1$ for integer $n = 21, 22, \ldots, 49$ and $\left\lfloor a+\frac{n}{50}\right\rfloor=0$ for integer $n = 1, 2, \ldots, 20$

$\left\lfloor a+\frac{20}{50}\right\rfloor=0\Rightarrow a+\frac{20}{50}<1\Rightarrow a<\frac{30}{50}$

$\left\lfloor a+\frac{21}{50}\right\rfloor=1\Rightarrow a+\frac{21}{50}\ge1\Rightarrow a\ge\frac{29}{50}$

$\Rightarrow\frac{29}{50}\le a<\frac{3}{5}$

$\Rightarrow 5.8\le 10a<6$

$\Rightarrow\lfloor 10a\rfloor=5$

7. Given two distinct positive integers a and b such that $(3b - 1)$ is divisible by $(2a + 1)$ and $(3a - 1)$ is divisible by $(2b + 1)$. Find the value of $a + b$.

【Answer】29

【Solution】

Suppose $\begin{matrix}3b-1=m\cdot(2a+1)\\3a-1=n\cdot(2b+1)\end{matrix}$ for some positive integer m and n

Then $9b-3=6am+3m$ ……(1)

$3a=2bn+n+1$ ……(2)

Sub (2) into (1), we obtain

$$9b-3=2m\cdot(2bn+n+1)+3m$$

$$9b-4mnb=2mn+5m+3$$

$$b=\frac{2mn+5m+3}{9-4mn}$$

Since b is a positive integer, $9-4mn>0\Rightarrow mn<2.25$

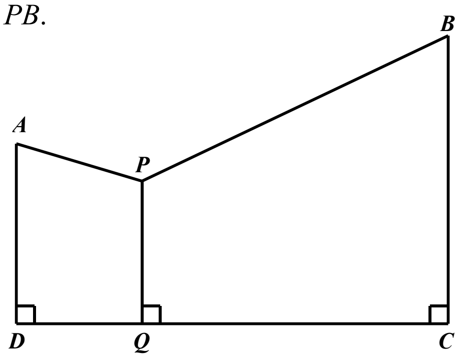
WLOG assume $m\le n$, then $(m,n) = (1,1)$ or $(1,2)$.

If $(m,n) = (1,1)$, $a = b$, which is a contradiction.

If $(m,n) = (1,2)$, $a = 17$, $b = 12$.

Hence $a+b = 29$.

8. As shown in the figure. If $\angle A = \angle B$, AD, PQ, BC are all perpendicular to CD, $AD = 10$, $PQ = 8$, $BC = 13$, $CD = 24$, find the value of $PA + PB$.



【Answer】25

【Solution】

Through point P, draw a line parallel to DC, intersecting AD and BC at E and F respectively.

$AE = 10 - 8 = 2$

$BF = 13 - 8 = 5$

Extend AP to point G on BC.

$\angle PGB = \angle A = \angle B$

$\Rightarrow PG = PB$

$\Rightarrow PA + PB = PA + PG = AG$

Through point G, draw a line parallel to DC, intersecting AD at point H.

In $\triangle AHG, AH^2 + HG^2 = AG^2$

$AH = AE + EH$ where $EH = FG = BF$

Hence $AH = AE + BF = 2 + 5 = 7$

Also, $AH = CD = 24$

Hence $AG = \sqrt{AH^2 + HG^2} = \sqrt{7^2 + 24^2} = 25$

$PA + PB = 25$

9. If real numbers x, y satisfy $x \geq y \geq 1$ and $2x^2 - 2xy - 6x + y^2 = -9$, find the value of $x + y$.

【Answer】6

【Solution】

$2x^2 - 2xy - 6x + y^2 = -9$

$(x^2 - 2xy + y^2) + (x^2 - 6x + 9) = 0$

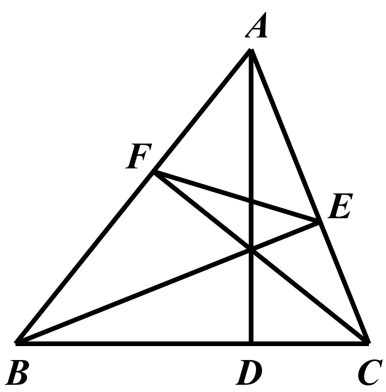
$(x - y)^2 + (x - 3)^2 = 0$

hence, $x = y = 3 \Rightarrow x + y = 6.$

10. As shown in the figure, AD, BE, CF are the three heights of triangle ABC, if $AB = 12$, $BC = 10$, $EF = 6$, find the length of BE.

【Answer】9.6

【Solution】



Since $BE \perp AC$ and $CF \perp AB$,

$\angle BEC = \angle BFC = 90^{\circ}$, so B, C, E, F are concyclic.

In this circle, $\angle FEB = \angle FCB = 90^{\circ} - \angle ABC$.

Also, $\angle AEF + \angle FEB = 90^{\circ}$,

Hence, $\angle AEF = \angle ABC$.

We have $\triangle AEF \sim \triangle ABC$.

$\frac{AE}{AB} = \frac{EF}{BC} \Rightarrow \frac{AE}{12} = \frac{6}{10} \Rightarrow AE = 7.2$

$\Rightarrow BE = \sqrt{AB^2 - AE^2} = \sqrt{12^2 - 7.2^2} = 9.6$

11. Find the number of positive integer pairs (x, y) that satisfy the following equation.

$$x\sqrt{y} + \sqrt{x}y - \sqrt{2026x} - \sqrt{2026y} + \sqrt{2026xy} = 2026$$

【Answer】4

【Solution】

Let $a = \sqrt{x}, b = \sqrt{y}, t = \sqrt{2026}$, then we have

$$a^2b + ab^2 - ta - tb + tab = t^2$$

$$(a^2b - ta) + (ab^2 - tb) + (tab - t^2) = 0$$

$$(ab - t)(a + b + t) = 0$$

Note a, b, t are all nonnegative, thus $a + b + t > 0 \Rightarrow ab - t = 0$

that is, $ab = t \Rightarrow \sqrt{xy} = \sqrt{2026} \Rightarrow xy = 2026$

Prime factorize 2026, we have

$2026 = 2 \times 1013$

Hence, there are 4 possible pairs (x, y).

12. If $m^2 = n + 5$, $n^2 = m + 5$, $m \neq n$, find the value of $m^5 - 2mn + n^5$.

【Answer】 –93

【Solution】

$m^2 = n + 5$……(1)

$n^2 = m + 5$……(2)

(1) – (2):

$$m^2 - n^2 = n - m$$
$$(m-n)(m+n+1) = 0$$
$$m+n+1 = 0$$
$$m+n = -1$$

$(1)\times(2)$:

$$m^2n^2 = (n+5)(m+5)$$
$$= mn + 5(m+n) + 25$$
$$= mn - 5 + 25$$
$$= mn + 20$$
$$m^2n^2 - mn - 20 = 0$$
$$(mn+4)(mn-5) = 0$$

Since $m+n=-1$, at least one of the two variables is negative.

Suppose $mn=5$, then both variables are negative,

from $m+n=-1$ we have

$|m|<1, |n|<1,$

$\Rightarrow mn<1$

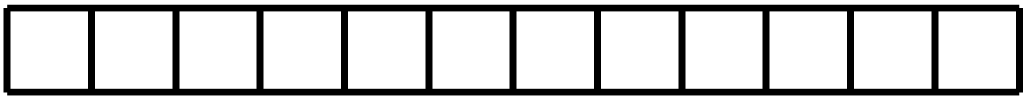
which is a contradiction.

Hence $mn \neq 5 \Rightarrow mn = -4$

$$m^5 = (m^2)^2 \cdot m$$
$$= (n+5)^2 \cdot m$$
$$= (n^2 + 10n + 25)m$$
$$= (m + 5 + 10n + 25)m$$
$$= (m + 1 + n + 9n + 29)m$$
$$= (9n + 29)m$$

$$m^5 - 2mn + n^5$$
$$= (9mn + 29n) - 2mn + (9mn + 29m)$$
$$= 16mn + 29(m+n)$$
$$= 16 \times (-4) - 29$$
$$= -93$$

13. Fill the 12 boxes shown in the diagram with consecutive positive integers 1, 2, 3, ... in ascending order. Each time you place a number, you must choose a box that is empty and whose adjacent boxes are also empty. Continue this process until no more numbers can be placed. How many different filling methods are there in total?



【Answer】7464

【Solution】

Name these 12 boxes from left to right, as box 1, 2, 3, …, 12.

The total number of boxes filled could be 4, 5 or 6.

(1) If only 4 boxes are filled, the only possible position is box 2, 5, 8, 11.

Then the number of different filling methods is $A_4^4 = 24$.

(2) If 5 boxes are filled, suppose they are box a_1, a_2, a_3, a_4, a_5, where $a_1 < a_2 < a_3 < a_4 < a_5$.

Then $a_2 - a_1, a_3 - a_2, a_4 - a_3, a_5 - a_4$ could only be 2 or 3; a_1 is 1 or 2, a_5 is 11 or 12.

That is, $a_1 - 1, a_2 - a_1 - 2, a_3 - a_2 - 2, a_4 - a_3 - 2, a_5 - a_4 - 2, 12 - a_5$, are either 0 or 1,

and the sum of these 6 terms is 3.

Number of different positions is $C_6^3 = 20$.

Then the number of different filling methods is $20 \times A_5^5 = 2400$.

(3) If 6 boxes are filled, suppose they are box $a_1, a_2, a_3, a_4, a_5, a_6$ ($a_1 < a_2 < a_3 < a_4 < a_5 < a_6$).

Like case (2), $a_1 - 1, a_2 - a_1 - 2, a_3 - a_2 - 2, a_4 - a_3 - 2, a_5 - a_4 - 2, a_6 - a_5 - 2, 12 - a_6$,

are either 0 or 1, and the sum of these 7 terms is 1.

Number of different positions is $C_7^1 = 7$.

Then the number of different filling methods is $7 \times A_6^6 = 5040$.

In all, the total number of different filling methods is $24 + 2400 + 5040 = 7464$.

14. Let t be a real number. If a and b are the two non-negative real roots of the quadratic equation $x^2 - 4x + t - 1 = 0$, find the minimum value of $(a^2 - 1)(b^2 - 1)$.

【Answer】-15

【Solution】

By Vieta's formulas

$a + b = 4$

$ab = t - 1$

There are two real roots.

$\Delta = 4^2 - 4(t-1) \geq 0$

$\Rightarrow t - 1 \leq 4 \Rightarrow t \leq 5$

Also, both roots are non-negative.

$ab = t - 1 \geq 0 \Rightarrow t \geq 1$

That is, $1 \leq t \leq 5$.

$$\begin{aligned} a^2 + b^2 &= (a+b)^2 - 2ab \\ &= 16 - 2(t-1) \\ &= 18 - 2t \end{aligned}$$

$$\begin{aligned} (a^2-1)(b^2-1) &= a^2b^2 - (a^2+b^2) + 1 \\ &= (t-1)^2 - (18-2t) + 1 \\ &= t^2 - 16 \end{aligned}$$

For $1 \leq t \leq 5$, this value increases as t increases.

Hence, when $t = 1$, the minimum value is attained, which is -15.

III. *Short Answer Questions(3) (16 marks)*

15. ***n*** balls numbered from **1** to ***n*** are placed into two boxes such that any two balls whose numbers add up to a perfect square must be placed in the same box. Each box must contain at least one ball. Find the maximum possible value of ***n*** that satisfies these conditions.

【Answer】13

【Solution】

Suppose the two boxes are box A and box B.

If $n \geq 14$, suppose ball 1 is in box A.

Then since $1+3=2^2, 1+8=3^2$, box A contains ball 3 and 8.

Then since $13+3=4^2, 13+12=5^2, 12+4=4^2$, box A contains ball 13, 12, 4.

Then since $4+5=3^2, 5+11=4^2, 11+14=5^2$, box A contains ball 5, 11, 14.

Then since $14+2=4^2, 2+7=3^2, 7+9=4^2$, box A contains ball 2, 7, 9.

Also $3+6=3^2, 6+10=4^2$, box A contains ball 6, 10.

Thus, box A already contains all 14 balls from ball 1 to ball 14.

And hence any ball with $n > 14$ is also in box A.

This implies Box B is empty, which does not satisfy the given conditions.

Hence $n < 14$.

Consider $n = 13$

suppose ball 1 is in box A.

Then since $1+3=2^2, 1+8=3^2$, box A contains ball 3 and 8.

Then since $13+3=4^2, 13+12=5^2, 12+4=4^2$, box A contains ball 13, 12, 4.

Then since $4+5=3^2, 5+11=4^2$, box A contains ball 5, 11.

Also $3+6=3^2, 6+10=4^2$, box A contains ball 6, 10.

So now Box A = {1,3,4,5,6,8,10,11,12,13} and Box B = {2,7,9}, both are nonempty.

Hence, 13 satisfies all given conditions.

$n_{\max} = 13$.