数学花园

探秘

MATHEMATICS GARDEN

2026 Spring Cup

Mathematical Olympiad

PRELIMINARY ROUND

Date： 31 January 2026

Time Given： 1 hour 30 minutes

Level: Secondary Advanced

Name: ________________

Parent’ s Phone Number: ________________

Instructions to Candidates

1. Do not open the booklet until you are told to do so.
2. Answer ALL 13 questions.
3. Write your answers in the answer sheet provided.
4. Questions 9(a), 12, 13 require full working steps.
5. Questions 1-4 are worth 8 marks each.
6. Questions 5-8 are worth 10 marks each.
7. Questions 9-11 are worth 14 marks each.
8. Questions 12-13 are worth 18 marks each.
9. No marks will be deducted for wrong answers.
10. No marks will be given for unanswered questions.
11. No calculators or mathematical instruments are allowed.

I. ***Short Answer Questions(1) (8 marks each, 32 marks in Total)***

1. Given that for any $x \in \mathrm{R}$, the function $f(x)$ satisfies $f\left(x+1\right)=\dfrac{f\left(x\right)-1}{f\left(x\right)+1}$, and $f(1) = -2$, find the value of $f(2026)$.

【Answer】 $\dfrac{1}{2}$

【Solution】

Since

$$f(x+4)=f((x+2)+2)$$
$$=\frac{f(x+2)-1}{f(x+2)+1}$$
$$=\frac{\frac{f(x)-1}{f(x)+1}-1}{\frac{f(x)-1}{f(x)+1}+1}$$
$$=-\frac{1}{f(x)},$$

we have

$$f\left(x+8\right)=f\left(x\right),$$

so the function $f(x)$ is periodic with period 8.

Therefore,

$$f\left(2013\right)=f\left(5\right)=-\frac{1}{f\left(1\right)}=\frac{1}{2}.$$

2. Let x and y be positive real numbers. What is the smallest possible value of $\dfrac{16}{x}+\dfrac{108}{y}+xy$?

【Answer】36

【Solution】

Using Arithmetic Mean $\geq$ Geometric Mean, we have

$$\frac{1}{3}\left(\frac{16}{x}+\frac{108}{y}+xy\right)\geq\sqrt[3]{\frac{16}{x}\times\frac{108}{y}\times xy}=12$$
$$\Rightarrow\frac{16}{x}+\frac{108}{y}+xy\geq 36$$

Equality is achieved when $\dfrac{16}{x}=\dfrac{108}{y}=xy=\dfrac{36}{3}=12$, which is satisfied when $x=\dfrac{4}{3}, y=9$. Thus, the smallest possible value is 36.

3. The result of the expression $\sqrt[3]{26+15\sqrt{3}}-\sqrt[3]{-26+15\sqrt{3}}$ is ________.

【Answer】4

【Solution】The given expression

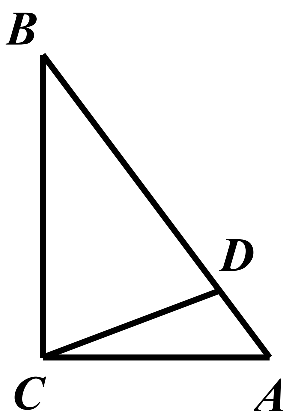
$$=\sqrt[3]{\left(\sqrt{3}\right)^3+3\cdot\left(\sqrt{3}\right)^2\cdot 2+3\cdot\left(\sqrt{3}\right)\cdot 2^2+2^3}-\sqrt[3]{\left(\sqrt{3}\right)^3-3\cdot\left(\sqrt{3}\right)^2\cdot 2+3\cdot\left(\sqrt{3}\right)\cdot 2^2-2^3}$$

$$=\sqrt[3]{\left(\sqrt{3}+2\right)^3}-\sqrt[3]{\left(\sqrt{3}-2\right)^3}=\sqrt{3}+2-\left(\sqrt{3}-2\right)=4$$

4. As shown in the figure, in a right triangle ABC, $\angle C = 90°$, $AC = 6$, $BC = 8$, point D is on the hypotenuse AB, and $\angle CDB = 2\angle B$, find the length of CD.

【Answer】5

【Solution】



Extend CD to point E such that $DE = DB$,

link BE, draw $DF \perp BE$ at point F,

and thus $\angle DBE=\angle DEB=\frac{1}{2}\angle CDB=\angle CBD$,

Given that $AC=6$, $BC=8$, we have $AB=10$,

$\frac{DE}{BE}=\frac{DE}{2EF}=\frac{1}{2\cos\angle DEB}=\frac{1}{2\cos\angle ABC}=\frac{AB}{2BC}=\frac{5}{8}$.

By the Angle bisector theorem, in $\triangle BCE$, $\frac{CD}{DE}=\frac{BC}{BE}$,

thus $\frac{CD}{BC}=\frac{DE}{BE}=\frac{5}{8}$, and we have $CD=\frac{5}{8}BC=5$.

II. ***Short Answer Questions(2) (10 marks each, 40 marks in Total)***

5. Find the units digit of $\left(2^1+1\right)\left(2^2+1\right)\left(2^3+1\right)...\left(2^{2017}+1\right)$.

【Answer】5

【Solution】

Since $\left(2^2+1\right)=5$, and for any positive integer $k, 2^k+1$ is an odd number, we have:

$\left(2^1+1\right)\left(2^2+1\right)\left(2^3+1\right)...\left(2^{2017}+1\right)\equiv 5(\text{mod}10)$.

6. Given that two real numbers x, y satisfy $20x^2 - 14x - 5 = 0$, $y^2 + 26y + 20 = 0$, and $xy \neq y + 1$, then find the value of $\frac{xy-y+x}{y}$.

【Answer】 $-\frac{5}{4}$

【Solution】

From the conditions given, we have

$20(x-1)^2+26(x-1)+1=0$， $20\left(\frac{1}{y}\right)^2+26\left(\frac{1}{y}\right)+1=0$， and $x-1\neq\frac{1}{y}$，

that is, $x-1$ and $\frac{1}{y}$ are two distinct roots of the equation $20t^2+26t+1=0$.

By Vieta's formulas, we have $x-1+\frac{1}{y}=-\frac{13}{10}$， $(x-1)\cdot\frac{1}{y}=\frac{1}{20}$，

and thus $\frac{xy-y+x}{y}=x-1+\frac{x-1}{y}+\frac{1}{y}=-\frac{13}{10}+\frac{1}{20}=-\frac{5}{4}$.

7. Find the value of sin6°sin42°sin66°sin78°.

【Answer】 $\frac{1}{16}$

【Solution】

Method 1:

Using trigonometric identities, we have:

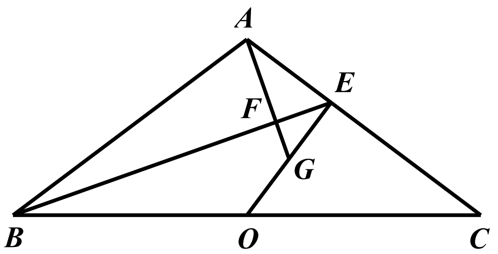
$$\begin{aligned}\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ &= \cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 84^\circ \\ &= -\cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 96^\circ \\ &= -\frac{\sin 192^\circ}{16\sin 12^\circ} \\ &= \frac{1}{16}\end{aligned}$$

Method 2:

Using the triple-angle formula, we have:

$$\begin{aligned}\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ &= \frac{4\cdot\sin 6^\circ \sin 54^\circ \sin 66^\circ}{4\cdot\sin 54^\circ}\cdot\frac{4\cdot\sin 18^\circ \sin 42^\circ \sin 78^\circ}{4\cdot\sin 18^\circ} \\ &= \frac{\sin 18^\circ}{4\cdot\sin 54^\circ}\cdot\frac{\sin 54^\circ}{4\cdot\sin 18^\circ} \\ &= \frac{1}{16}\end{aligned}$$

8. As shown in the figure, in an isosceles triangle ABC, $AB = AC$, $BC = 20$, point O is the midpoint of BC, $OE \perp AC$ at point E, $AF \perp BE$ at point F and intersects OE at point G, $OG = 3$. Find the area of triangle $\triangle ABC$.



【Answer】75

【Solution】

Link AO intersecting BE at point D,

let $OA = x$,

as $AB = AC$, and O is the midpoint of BC,

thus $OA \perp BC$, $OC = \frac{1}{2}BC = 10$,

we have $CA = \sqrt{OC^2 + OA^2} = \sqrt{100 + x^2}$,

Also, $OE \perp AC$, by the geometric mean theorem, we have $CE \cdot CA = OC^2$,

thus $CE = \frac{100}{\sqrt{100 + x^2}}$,

Also, $AF \perp BE$, thus $\angle OAG = 90° - \angle ADF = 90° - \angle BDO = \angle CBE$,

and $\angle OGA = \angle GFE + \angle FEG = 90° + \angle FEG = \angle OEC + \angle FEG = \angle CEB$,

hence $\triangle OGA \backsim \triangle CEB$, thus $\frac{OG}{CE} = \frac{OA}{CB}$, we have $\frac{3\sqrt{100 + x^2}}{100} = \frac{x}{20}$,

thus $9\left(100 + x^2\right) = 25x^2$, $x^2 = \frac{225}{4}$, we find $x = \frac{15}{2}$, thus $S_{\triangle ABC} = \frac{1}{2}BC \cdot OA = 75$.

III. ***Short Answer Questions(3) (14 marks each, 42 marks in Total)***

9. On the six faces of a cube, each face is labeled with a distinct positive integer.
   The product of the numbers on two adjacent faces is written on their common edge, and the product of the numbers on the three faces meeting at a vertex is written at that vertex.
   Among these 26 numbers written,
   (a) At most how many different values can appear among these 26 numbers? **(7 marks)**

   (b) At least how many different values must appear? Please explain your reasoning. **(7 marks)**

【Answer】(1) 26; (2) 10

【Solution】

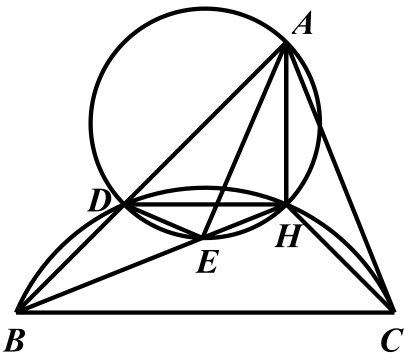
(1) The maximum possible answer is 26, so give an example that achieves this maximum.
Suppose each of the six faces is labeled with: $2,3,5,7,11,13$.
Then the 26 numbers written are distinct, hence answer is 26.

(2) Suppose each of the six faces is labeled with distinct positive integers,
a_1,a_2,a_3,a_4,a_5,a_6, without loss of generality, assume $a_1<a_2<a_3<a_4<a_5<a_6$.
WLOG, assume the four faces adjacent to the face labeled a_6 are labeled
b_1,b_2,b_3,b_4 ($b_1<b_2<b_3<b_4$),
also assume among these four faces, the two faces adjacent to the face labeled b_4 are labeled
c_1,c_2 ($c_1<c_2$),
then among the 26 numbers written, we have

$$a_1<a_2<a_3<a_4<a_5<a_6<a_6b_2<a_6b_3<a_6b_4<a_6b_4c_2,$$

and thus, there are at least 10 distinct values.
The minimum possible answer is 10, so give an example that achieves this minimum.
Suppose on a cube, we label the top and bottom faces with $1,2$, the left and right faces with $2^2,2^3$, and the front and back faces with $2^4,2^5$.
In this way all the 26 numbers written are exactly $2^0,2^1,2^2,\cdots,2^9$.
Hence, there are exactly 10 distinct values.
Hence answer is 10.

10. In an acute triangle ABC, $\angle ABC = 45°$ and H is the orthocenter. A circle passing through points B, H, and C intersects AB at point D. Another circle passing through points A, D, and H intersects BH at point E. Given that $DE = 2$ and $EH = 3\sqrt{2}$, find the length of EA.



【Answer】8

【Solution】

As H is the orthocenter, we have

$\angle BCH = \angle BAH = 90° - \angle ABC = 45°$,

From the conditions given，

B , C , H , D are concyclic,

thus $\angle ADH = \angle BCH = 90° - \angle ABC = 45°$,

hence $\angle AHD = 90°$，we have $AH = DH = \frac{\sqrt{2}}{2}AD$,

Also A , D , E , H are concyclic,

According to Ptolemy's theorem， $EA \cdot DH = AD \cdot EH + AH \cdot DE$,

hence $EA = \sqrt{2}EH + DE = 8$.

11. The inequality $a^2 + 4b^2 + 2n - 2025 \le 2ab + 3a + 6b$ (where n is a positive integer) has exactly 16 integer solutions. What is the value of n?

【Answer】1013

【Solution】

By completing the square, we have

$$\left(a-b-\frac{3}{2}\right)^2+3\left(b-\frac{3}{2}\right)^2\le 2034-2n,$$

that is, $(2a-2b-3)^2+3(2b-3)^2\le 8136-8n$.

Let $2a-2b-3=x$, $2b-3=y$.

If a,b are integers, then x,y are odd.

Conversely, if x,y are odd, then $b=\frac{y+3}{2}$, $a=\frac{x+y+6}{2}$, we have a,b are integers.

Hence the inequality $x^2+3y^2\le 8136-8n$ has exactly 16 odd integer solutions.

If (x_0,y_0) is one of the solutions,

then $(x_0,-y_0)$, $(-x_0,y_0)$, $(-x_0,-y_0)$ must also be solutions to the inequality.

Hence, the inequality has exactly 4 positive odd integer solutions.

Discuss by cases:

When $y=1$,

$x=1$, $x^2+3y^2=4$;

$x=3$, $x^2+3y^2=12$;

$x=5$, $x^2+3y^2=28$;

$x\ge 7$, $x^2+3y^2\ge 52$;

When $y=3$,

$x=1$, $x^2+3y^2=28$;

$x\ge 3$, $x^2+3y^2\ge 36$;

When $y\ge 5$, $x^2+3y^2\ge 76$;

Thus $28\le 8136-8n<36$,

that is $\frac{2025}{2}\le n<\frac{2027}{2}$,

also n is a positive integer,

hence $n=1013$.

IV. ***Long Answer Questions (18 marks each, 36 marks in Total)***

12. Find all non-negative integer solutions (x, y, z) to the equation $2^x - 5^y \cdot 7^z = 1$.

【Answer】$(x, y, z) = (1, 0, 0), (3, 0, 1)$

Solution

We first reduce the equation modulo 3:

$$(-1)^x - (-1)^y \cdot 1^z \equiv 1 \pmod 3$$

This implies x is odd and y is even. Let $x = 2m + 1,\ y = 2n$ $(m, n \in \mathbb{N})$. Substituting into the original equation:

$$2 \cdot 4^m - 25^n \cdot 7^z = 1$$

If $n \neq 0$, reduce modulo 5:

$$2 \cdot (-1)^m \equiv 1 \pmod 5$$

This is a contradiction, so $n = 0$, and the equation becomes:

$$2 \cdot 4^m - 7^z = 1$$

1. **If $z = 0$:**

   $2 \cdot 4^m = 2 \implies m = 0$, giving $(x, y, z) = (1, 0, 0)$.

2. **If $z \neq 0$:**

   Reduce modulo 4:

   $$-(-1)^z \equiv 1 \pmod 4$$

   So z is odd. Let $z = 2p + 1$ $(p \in \mathbb{N})$:

   $$2 \cdot 4^m - 7 \cdot 49^p = 1$$

   - If $p = 0$: $2 \cdot 4^m = 8 \implies m = 1$, giving $(x, y, z) = (3, 0, 1)$.
   - If $p \neq 0$: $m \geq 4$. Reduce modulo 16:

     $$-7 \cdot 1^p \equiv 1 \pmod{16}$$

     This is a contradiction.

Conclusion

All non-negative integer solutions are $(x, y, z) = (1, 0, 0),\ (3, 0, 1)$.

13. Prove that if p is a prime number, then $p^2 \mid \left(\binom{2p}{p} - 2\right)$.

Solution

We use the well-known combinatorial identity (considering the coefficient of x^n in the expansion of $(1+x)^n \cdot (1+x)^n$):

$$\binom{m+n}{n} = \binom{m}{0}\binom{n}{n} + \binom{m}{1}\binom{n}{n-1} + \cdots + \binom{m}{n}\binom{n}{0}$$

Set $m = n = p$, so:

$$\binom{2p}{p} - 2 = \left(\binom{p}{1}\right)^2 + \cdots + \left(\binom{p}{p-1}\right)^2$$

Since $\binom{p}{k} \in \mathbb{Z}$, we know $k! \mid p(p-1)(p-2)\ldots(p-k+1)$ for $k = 1, 2, \ldots, p-1$.

As $k < p$ and p is prime, $k!$ and p are coprime, so:

$$k! \mid (p-1)(p-2)\ldots(p-k+1)$$

This implies $\frac{(p-1)!}{k!(p-k)!} \in \mathbb{Z}$, so $\binom{p}{k} = p \cdot \frac{(p-1)!}{k!(p-k)!}$ is divisible by p.

Thus each $\binom{p}{k}$ $(1 \le k \le p-1)$ is a multiple of p, so $\left(\binom{p}{k}\right)^2$ is a multiple of p^2.

Summing these terms shows $\binom{2p}{p} - 2$ is a sum of multiples of p^2, hence $p^2 \mid \left(\binom{2p}{p} - 2\right)$.

Note

Starting from -2 and using the combinatorial identity simplifies the problem and connects it to squares.