数学花园

探秘

MATHEMATICS GARDEN

2023 Spring Cup

Mathematical Olympiad

PRELIMINARY ROUND

Date : 28 January 2023

Time Given : 1 hour 30 minutes

Level: Secondary 1

Name: ____________

Instruction to Candidates

1. Do not open the booklet until you are told to do so.
2. Answer ALL 18 questions.
3. Write your answers in the answer sheet provided
4. No steps are needed to justify your answers.
5. Questions 1-4 are worth 5 marks each.
6. Questions 5-11 are worth 6 marks each.
7. Questions 12-17 are worth 8 marks each.
8. Questions 18 are worth 10 marks.
9. No marks will be deducted for wrong answers.
10. No marks will be given for unanswered questions.
11. No calculators or mathematical instruments are allowed.

<u>Questions 1 to 4 are worth 5 marks each.</u>

1. Let a and b be positive number satisfying $a \cdot b^2 = 3$ and $a^4 \cdot b^5 = 6$. Find the value of $a^7 \cdot b^8$.

【Solution】 $a^3 \cdot b^3 = \left(a^4 \cdot b^5\right) \div \left(a \cdot b^2\right) = 6 \div 3 = 2$, $a^7 \cdot b^8 = \left(a^4 \cdot b^5\right) \cdot \left(a^3 \cdot b^3\right) = 6 \times 2 = 12$.

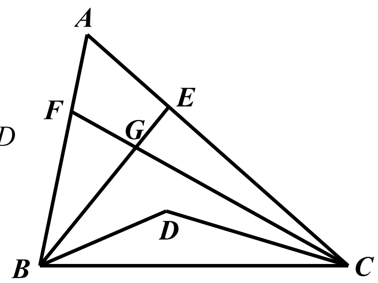
2. What are the last four digits of the sum

$$1+12+123+1234+12345+123456+1234567+12345678+123456789?$$

Give your answer as a 4-digit number.

【Solution】 $1+2+3+4+5+6+7+8+9=45$, so unit digit is 5;
$1+2+3+4+5+6+7+8+4=40$, so 10s digit is 0;
$1+2+3+4+5+6+7+4=32$, so 10s digit is 2;
$1+2+3+4+5+6+3=24$, so 10s digit is 4.
So the answer is 4205.

3. As shown in the figure, BE is the bisector of $\angle ABD$ and CF is the bisector of $\angle ACD$. If BE and CF intersect at G. If $\angle BDC = 140°$ and $\angle BGC = 100°$, what is the degree of $\angle A$?



【Solution】 $\because \angle BDC = \angle BGC + \angle DBE + \angle DCF = 140°$
$\therefore \angle DBE + \angle DCF = 140° - \angle BGC = 40°$
$\because$ BE is the bisector of $\angle ABD$ and CF is the bisector of $\angle ACD$
$\therefore \angle ABE + \angle ACF = \angle DBE + \angle DCF = 40°$
$\therefore \angle A = \angle BGC - \left(\angle ABE + \angle ACF\right) = 60°$

4. Let n be a positive integer. If the highest common factor of n and 140 is 14, and the highest common factor of n and 66 is 6, what is the sum of the two smallest positive values of n?

【Solution】 $140 = 2^2 \times 5 \times 7$ and $14 = 2 \times 7$, so n is a multiple of 2 and 7 but not a multiple of 2^2 and 5. $66 = 2 \times 3 \times 11$ and $6 = 2 \times 3$, so n is a multiple of 2 and 3 but not a multiple of 11. Then the smallest possible value of n is $2 \times 3 \times 7 = 42$, and the secend smallest possible value of n is $2 \times 3^2 \times 7 = 126$. The sum is $42 + 126 = 168$.

Questions 5 to 11 are worth 6 marks each.

5. Let m and n be two positive integers satisfying

$$\frac{5}{6}-\frac{7}{12}+\frac{9}{20}-\frac{11}{30}+\frac{13}{42}-\frac{15}{56}+\frac{17}{72}=\frac{m}{n}.$$

Find the minimum value of $m+n$.

【Solution】

$$\frac{5}{6}-\frac{7}{12}+\frac{9}{20}-\frac{11}{30}+\frac{13}{42}-\frac{15}{56}+\frac{17}{72}$$

$$=\frac{2+3}{2\times3}-\frac{3+4}{3\times4}+\frac{4+5}{4\times5}-\frac{5+6}{5\times6}+\frac{6+7}{6\times7}-\frac{7+8}{7\times8}+\frac{8+9}{8\times9}$$

$$=\left(\frac{1}{2}+\frac{1}{3}\right)-\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}\right)-\left(\frac{1}{5}+\frac{1}{6}\right)+\left(\frac{1}{6}+\frac{1}{7}\right)-\left(\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{8}+\frac{1}{9}\right)$$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{3}-\frac{1}{4}+\frac{1}{4}+\frac{1}{5}-\frac{1}{5}-\frac{1}{6}+\frac{1}{6}+\frac{1}{7}-\frac{1}{7}-\frac{1}{8}+\frac{1}{8}+\frac{1}{9}$$

$$=\frac{1}{2}+\frac{1}{9}$$

$$=\frac{11}{18}$$

$m+n=29$

6. If a, b and c are integers satisfying $abc=240$, $ac+b=46$ and $a+bc=64$. Find the value of $a+b+c$.

【Solution】∵ $ac+b=46$

∴ $abc+b^2=46b$, $b^2-46b+240=0$

So $b=40$ or $b=6$.

∵ $a+bc=64$

∴ $a^2+abc=64a$, $a^2-64a+240=0$

So $a=60$ or $a=4$.

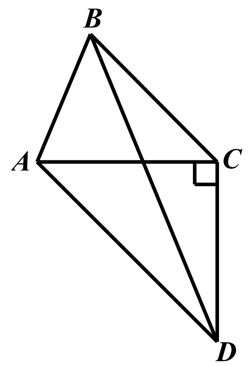
If $b=40$, $ac=6=1\times6=2\times3$ and $a+40c=64$. a and c can't be integers at the same time.

Likewise, $a=60$ is also impossible.

So $b=6$ and $a=4$.

Then $c=10$, $a+b+c=20$

7. In the quadrilateral *ABCD*, $\triangle ABC$ and $\triangle ACD$ are both isosceles triangles and point *C* is the common apex. Given that $\angle ACD = 90°$, $\angle ACB = 45°$ and $BD = 12$. Find the area of quadrilateral *ABCD*.



【Solution】

Draw a line *BP* such that $BP \perp AC$.

The area of $\triangle ABC = \frac{1}{2} \times AC \times BP$

The area of $\triangle ACD = \frac{1}{2} \times AC \times CD$

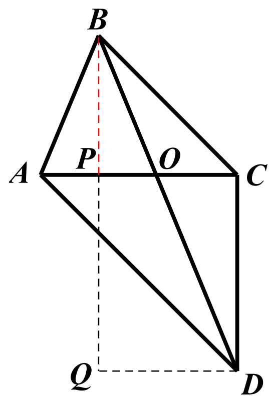
Let $AC = CD = BC = a$.

In $\triangle BPC$, $BP^2 + CP^2 = 2BP^2 = BC^2 = a^2$

$\therefore BP = CP = \frac{\sqrt{2}}{2}a$

Extend *BP* to *Q*, draw a line *DQ* such that $DQ \perp BQ$

So $DQ = CP = \frac{\sqrt{2}}{2}a$, $PQ = CD = a$, $BQ = BP + PQ = \left(1 + \frac{\sqrt{2}}{2}\right)a = \frac{2+\sqrt{2}}{2}a$



In $\triangle BDQ$, $BQ^2 + DQ^2 = BD^2$

$$\left(\frac{2+\sqrt{2}}{2}a\right)^2 + \left(\frac{\sqrt{2}}{2}a\right)^2 = 12^2$$

$\therefore a^2 = \frac{144}{2+\sqrt{2}}$

The area of *ABCD*:

$$\begin{aligned}\frac{1}{2} \times AC \times BP + \frac{1}{2} \times AC \times CD &= \frac{1}{2} \times AC \times (BP + CD) \\ &= \frac{1}{2}a\left(\frac{\sqrt{2}}{2}a + a\right) \\ &= \frac{2+\sqrt{2}}{4}a^2 \\ &= 36\end{aligned}$$

8. If $(4x-1)^4 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, find the value of $a_0 - \frac{a_1}{2} + \frac{a_2}{2^2} - \frac{a_3}{2^3} + \frac{a_4}{2^4}$.

【Solution】Let $x = -\frac{1}{2}$, $a_0 - \frac{a_1}{2} + \frac{a_2}{2^2} - \frac{a_3}{2^3} + \frac{a_4}{2^4} = \left[4\times\left(-\frac{1}{2}\right)-1\right]^4 = (-3)^4 = 81$

9. If $x - z = y - w = 26$, find the minimum value of $x^2 + y^2 + z^2 + w^2 - xy - yz - zw - wx$.

【Solution】$\because x - z = y - w = 26$
$\therefore x = 26 + z$ and $y = 26 + w$

Then $x^2 + y^2 + z^2 + w^2 - xy - yz - zw - wx$

$$= (26+z)^2 + (26+w)^2 + z^2 + w^2 - (26+z)(26+w) - (26+w)z - zw - w(26+z)$$

$$= 26^2 + 2z^2 + 2w^2 - 4zw = 26^2 + 2(z-w)^2 \geq 26^2 = 676.$$

When $z = w$ and $x = y = w + 26$, the minimum value of $x^2 + y^2 + z^2 + w^2 - xy - yz - zw - wx$ is 676.

10. How many integers k are there such that the equation

$$\frac{(k+4)x^2 + (k+2)x - 8}{(k+1)x - 2} = kx + 1$$

has real number solutions?

【Solution】$(k^2-4)x^2 - (2k+1)x + 6 = 0$

If the equation has real number solutions, $\Delta = (2k+1)^2 - 24(k^2-4) = -20k^2 + 4k + 97 \geq 0$.

The possible values of integers k are $-2, -1, 0, 1, 2$.
When $k = -2$, $x = -2$ (extraneous root);

When $k = -1$, $x = \frac{1 \pm \sqrt{73}}{6}$;

When $k = 0$, $x = \frac{-1 \pm \sqrt{97}}{8}$;

When $k = 1$, $x = 1$ (extraneous root) or $x = -2$;

When $k = 2$, $x = \frac{6}{5}$.

So there are 4 possible value of k.

11. There are nine robots numbered 1~9 standing in the 3 × 3 grid table, and each robot says one sentence. It is found that exactly one robot in each row tells lies, and exactly one robot in each column tells lies. Find the 3-digit number $\overline{def}$.

a: The numbers of the three robots in my column can be appropriately adjusted to form an arithmetic sequence;
b: The difference between the numbers of my left and right robots is 1;
c: I'm number 5;
d: There are two robots in my row telling lies;
e: My number is the smallest in my row;
f: The sum of the numbers of robots in my row is 21;
g: The sum of the numbers of two robots adjacent to me is 12;
h: The product of the three number of robots in my row is 6;
i: The numbers in my column are all even numbers.

a	***b***	***c***
d	***e***	***f***
g	***h***	***i***

【Solution】The robot in row 2 column 1 must tell lies, and there will be only 2 cases for the other two robots who tell lies.
Case 1:
If the robots who tell lies is the second in the first row and the third in the third row.
From the second in the third row, we can know that the numbers in this row is 1, 2, 3. From the first in the third row, we can know that the right one of it is 3, and the top side is 9. Since the sum of numbers in the second row is21, we can know that the number in the second row is 4, 8 and 9. However, at this time, the first column is no longer possible to become an arithmetic sequence.
Case 2:
So the other two robots who tell lies must be the third in the first row and the second in the third row. If there is no number 9 in the second row, it must be 6+7+8=21, and the order can only be 7, 6, 8. From the left bottom corner, we can know that the second in the third row is 5. Then the first column can only be 1, 4, and 7, but at this time, 4 and 6 are both in the first two columns, which are in contradiction with the condition that the numbers in the third column are all integer (there are only 4 even numbers in total). The number 9 is in the second row, and it can only be in the first column, so the second in the third row is 3. From the fact that the third in the second row is an even number, we can know that the second row can only be 9+4+8=21.
Ans：948

<u>Questions 12 to 17 are worth 8 marks each.</u>

12. Find the value of $\sqrt{3(\sqrt{2}+\sqrt{3}+\sqrt{5})(3\sqrt{2}+2\sqrt{3}-\sqrt{30})}$.

【Solution】

$$\sqrt{3(\sqrt{2}+\sqrt{3}+\sqrt{5})(3\sqrt{2}+2\sqrt{3}-\sqrt{30})}$$
$$=\sqrt{3(\sqrt{2}+\sqrt{3}+\sqrt{5})\sqrt{6}(\sqrt{3}+\sqrt{2}-\sqrt{5})}$$
$$=\sqrt{3\sqrt{6}\left[(\sqrt{2}+\sqrt{3})^2-5\right]}$$
$$=6$$

13. Given that $a^2+4a+1=0$ and $\frac{a^4+ma^2+1}{2a^3+ma^2+2a}=-10$, find the value of m.

【Solution】Since $a^2+4a+1=0$

Hence $a+\frac{1}{a}=-4$, $a^2+\frac{1}{a^2}=14$.

Then $\frac{a^4+ma^2+1}{2a^3+ma^2+2a}=\frac{a^2+m+\frac{1}{a^2}}{2a+m+\frac{2}{a}}=\frac{14+m}{m-8}=-10$, $m=6$.

14. Let a, b and c be rational numbers satisfying $\begin{cases} a(b+c)+1=bc \\ b(c+a)-7=2ca \\ c(a+b)-4=4ab \end{cases}$. Find the value of $a^2+b^2+c^2$.

【Solution】

If $\begin{cases} a(b+c)+1=bc \\ b(c+a)-7=2ca \\ c(a+b)-4=4ab \end{cases}$, $\begin{cases} ab+ac+1=bc \\ ab+bc-7=2ac \\ ac+bc-4=4ab \end{cases}$.

Then $\begin{cases} ab=15 & ① \\ ac=24 & ② \\ bc=40 & ③ \end{cases}$

①×②: $a^2bc=360$ ④

④÷③: $a^2=9$

①×③: $ab^2c=600$ ⑤

⑤÷②: $b^2=25$

②×③: $abc^2=960$ ⑥

⑥÷①: $c^2=64$

$a^2+b^2+c^2=9+25+64=98$

15. If n is an integer satisfying the value of $n^3-14n^2+56n-64$ is a prime number, find the value of n.

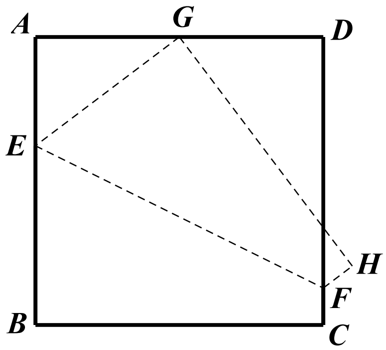
【Solution】

Factorize: $n^3-14n^2+56n-64=(n-8)(n-4)(n-2)$

Since $(n-8)(n-4)(n-2)$ is a prime number,

then the absolute value of two factors of it must be 1.
Hence one of factors must be 1 and another one of factors must be -1.
So $n=3$.

16. Let $ABCD$ be a square sheet of paper with $AB = 12$. If we fold the sheet of paper along the line EF, point B will coincide with G, the midpoint of AD as shown in the diagram. Find the area of the quadrilateral $BCFE$.



【Solution】 $AG=6$
Let $AE=x$, $EG=BE=12-x$.
∵ $AE^2+AG^2=EG^2$

∴ $x^2+36=(12-x)^2$

∴ $x=\frac{9}{2}$
Connect B and G, $BG\perp EF$.
∵In a square, the line segments perpendicular to each other must be equal.

∴ $EF=BG=\sqrt{6^2+12^2}=6\sqrt{5}$

Make $FM\perp AB$ at point M through F.

In $\triangle EFM$, $EM=\sqrt{\left(6\sqrt{5}\right)^2-12^2}=6$

∴ $BM=CF=1.5$

So the area of the quadrilateral $BCFE$ is $\frac{1}{2}(1.5+7.5)\times 12=54$.

17. If p and q are two prime numbers such that p^2-4q^4 is a square number, how many such pairs of (p,q) can be found?

【Solution】Let $p^2-4q^4=k^2(k\geq 0)$

Then $(p+k)(p-k)=4q^4$.

Since $p+k\geq p-k$, $p+k>0$, $p+k$ and $p-k$ are both odd or even.

Hence there will be 3 cases:

$\begin{cases} p+k=2q^4 \\ p-k=2 \end{cases}$, $\begin{cases} p+k=2q^3 \\ p-k=2q \end{cases}$ and $\begin{cases} p+k=2q^2 \\ p-k=2q^2 \end{cases}$

Case 1: $\begin{cases} p+k=2q^4 \\ p-k=2 \end{cases}$

Hence $p=q^4+1$. At this time, p and q must be an odd number and an even number. So $q=2$ and $p=17$.

Case 2: $\begin{cases} p+k=2q^3 \\ p-k=2q \end{cases}$

Hence $p=q(q^2+1)$.

But p and q are both prime numbers, it's impossible.

Case 3: $\begin{cases} p+k=2q^2 \\ p-k=2q^2 \end{cases}$

Hence $p=2q^2$. But p and q are both prime numbers, it's impossible.

There is only 1 pair.