# 2023 Spring Cup
# Mathematical Olympiad
## PRELIMINARY ROUND

Date: 28January 2023

Time Given: 1 hour 30 minutes

Level: Primary 5

Name: _____

## Instructions to Candidates

1. Do not open the booklet until you are told to do so.
2. Answer ALL 20 questions.
3. Write your answers in the answer sheet provided.
4. No steps are needed to justify your answers.
5. Questions 1-7 are worth 4 marks each.
6. Questions 8-14 are worth 6 marks each.
7. Questions 15-19 are worth 8 marks each.
8. Questions 20 is worth 10 marks.
9. No marks will be deducted for wrong answers.
10. No marks will be given for unanswered questions.
11. No calculators or mathematical instruments are allowed.

Questions 1 to 7 are worth 4 marks each.

1. Calculate: $\frac{13}{35}\times 5+\frac{13}{35}\times 6+\frac{13}{35}\times 7+\frac{13}{35}\times 8+\frac{13}{35}\times 9$

【Solution】 $\frac{13}{35}\times(5+6+\cdots+9)=35\times\frac{13}{35}=13$

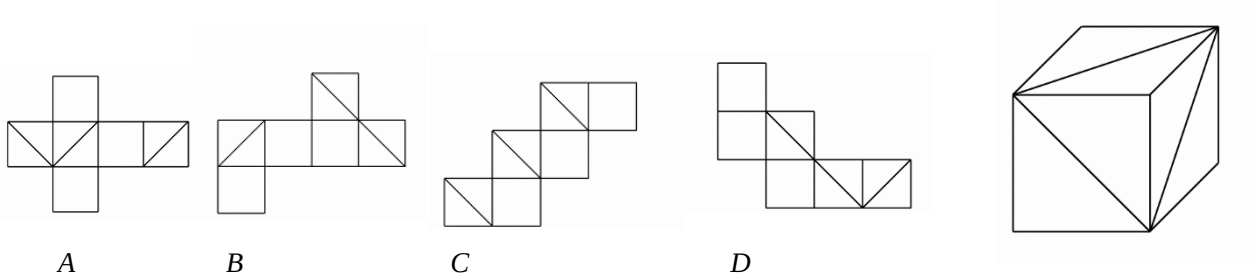
2. The sum of 7 natural numbers is 210.By arranging them in ascending order, the difference between each pair of adjacent numbers is 5, what is the 6th number?

【Solution】 The question implies that the 7 numbers form an arithmetic sequence. Hence the number in the middle should be 210÷7=30. As a result, these 7 numbers are 15, 20, 25, 30, 35, 40, 45 and the 6th number is 40.

3. It takes 225 digits to number the pages of a book, how many pages are there in this book?

【Solution】
Page 1 to 9 use $1\times 9=9$ digits.
Page 10 to 99 use $2\times 90=180$ digits.
Page 100 to 999 use $3\times 900=2700$ digits.
Since $9+180<225<9+180+2700$, the last page's number falls within 100-999. The number of all 3-digit page numbers is $225-9-180=36$, and each page number starting from 100 uses 3 digits, therefore there are $36\div 3=12$ pages after page 99. 99+12=111. There are 111 pages in this book.

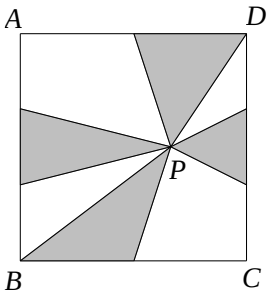
4. The "net" of a three-dimensional shape refers to its two-dimensional figure when laid out flat. Which of the following "net" corresponds to the cube on the right?



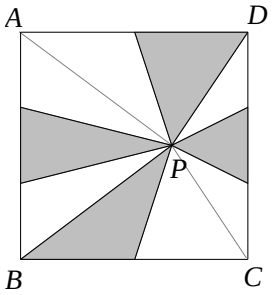
*A* *B* *C* *D*

【Solution】
Firstly D does not form a cube at all. D is wrong.
Secondly, in the 3-D shape shown on the right, the three sides where the diagonal is drawn are adjacent to each other, whereas A makes two sides containing the diagonal opposite each other. A is wrong.
Thirdly, any two of the three diagonals should intersect each other at a different point, forming a triangle, whereas in C, all the three diagonals intersect at one point, forming no triangle at all. C is wrong.
Finally, B can form the required 3-D shape.
The answer is B.

5. As shown in the figure, the square $ABCD$ has side 6cm and point $P$ is a point lying in this square. If the sides of the square $ABCD$ are divided into two or three segments evenly such that they can form triangles with the point $P$, find the area of the shaded region.




【Solution】
First, link PA, PC.

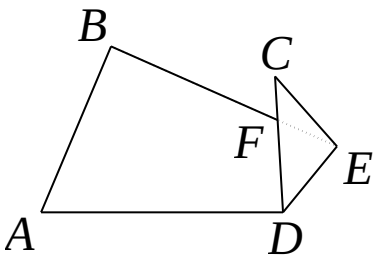



Because the combined area of $\Delta PAD$ and $\Delta PBC$ is equal to half of that of square ABCD, the combined area of the two shaded triangles on top and bottom is equal to one fourth of ABCD. Similarly, the combined area of the two shaded triangles on left and right is equal to one sixth of ABCD. Hence the overall area of the shade region is $6^2 \times (\frac{1}{4}+\frac{1}{6}) = 15$ cm².

6. Worker A takes 20 days to finish a job. Worker B takes 15 days to finish the same job. If worker A does the job for a few days before worker B takes over the job, it takes 18 days to finish the job. How many days does worker A and B work in this job respectively?

【Solution】View the job as "1". Per day worker A finishes $\frac{1}{20}$ while worker B finishes $\frac{1}{15}$. Assume that only worker A works for the entire 18 days, $\frac{1}{20}\times 18 = \frac{9}{10}$, the missing part is $1-\frac{9}{10}=\frac{1}{10}$. Hence during these 18 days, worker B works for $\frac{1}{10}\div\left(\frac{1}{15}-\frac{1}{20}\right)=6$ days, and worker A works for the other $18-6=12$ days. The answer is 12 and 6 days.

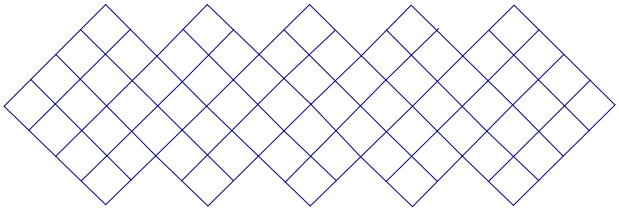
7. A triangle is folded along the line *DE* as shown below. Given that $\angle B = 74°$, $\angle A = 70°$ and $\angle CEB = 20°$, what is $\angle ADC$?




【Solution】$\angle C$=180° – 70° – 74°=36°, $\angle DFE$=36° + 20°=56°, $\angle BFD$=180° – 56°=124°, $\angle ADC$=360° – 70° – 74° – 124°=92°.

Questions 8 to 14 are worth 6 marks each.

8. How many squares can be found in the figure below?



【Solution】Within each 4×4 square, there are 4 types of squares, including $4^2$ squares of side length 1, $3^2$ squares of side length 2, $2^2$ squares of side length 3 and $1^2$ squares of side length 1. This means a total of $4^2+3^2+2^2+1^2=30$ squares in each 4×4 square. Two adjacent 4×4 squares have an overlapping region, which is a $2\times 2$ square. In each overlapping region, there are 5 squares. There are altogether five 4×4 square and four overlapping regions. Hence, the total number of squares in this figure is $30\times 5-5\times 4=130$.

9. Amelia and Benjamin want to purchase an item. According to the price tag, Amelia is short of 40 dollars and Benjamin is short of $\frac{1}{4}$ of the price. After negotiating, they received a 10% discount for the item, and they managed to make the purchase by combining their money. If they are left with 28 dollars, what was the original price of the item?

【Solution】On one hand, Amelia is short of 40 dollars while Benjamin only has $\frac{3}{4}$ of the price. On the other hand, their money combined is more than $\frac{9}{10}$ of the original price by 28 dollars. If they managed to get 40 dollars more, their money combined would be equal to $(1+\frac{3}{4})$ of the original price, and also equal to $\frac{9}{10}$ of the original price plus an extra (40+28) dollars. Therefore, the original price is, $(40+28)\div\left(1+\frac{3}{4}-\frac{9}{10}\right)=80$ dollars.

10. The ratio between red balls and white balls in a bag is $19:13$. First after adding in some red balls into the bag, the ratio becomes $5:3$. Thenafter adding in some white balls, the ratio becomes $13:11$. Given that the number of red balls added is 80 less than the number of white balls added, how many balls are there in the beginning?

【Solution】The first time only red balls are being added, so white balls remain unchanged. We can unify the units of the white balls, $19:13=57:39$, $5:3=65:39$.
The second time only white balls are being added, so before and after this action, red balls remain unchanged. We can unify the units of the red balls, $13:11=65:55$.
After the two steps, red balls increase by 65-57=8 units while white balls increase by 55-39=16 units. The increase in red balls is 16-8=8 units less than the increase in white balls. 8 units are equal to 80 balls. Therefore, one unit is 10 balls.
Originally there are $10\times(57+39)=960$ balls in total.

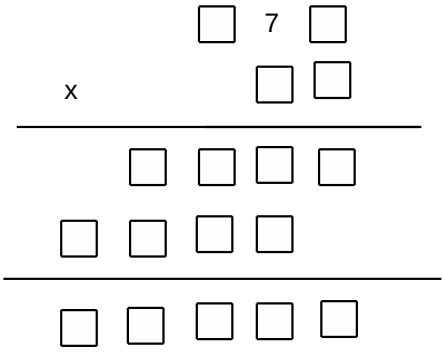
11. It takes a train 52 seconds to go through a 320 meters long bridge. If the train speeds up by $\frac{1}{4}$ of its original speed, it can go across a bridge of 864 meters within 1 minutes and 36 seconds. Find the speed of the train when it goes across the bridge and the length of the train.

【Solution】 After speeding up, it takes 96 seconds. So if moving at the original speed, the train would take $96\times(1+\frac{1}{4})$ =120 seconds to go across the 864m bridge. The speed of the train is $(864-320)\div(120-52)$ =8 m/s, and therefore the length of the train is $52\times8-320$=96m.

12. A total of 362880 distinct 9-digit numbers can be formed by using the number 1 to 9 without repetition. Find the greatest common divisor of the 362880 numbers.

【Solution】Firstly, $1+2+\cdots+9=45$ is a multiple of 9, and hence 9 is a common divisor for all these 9-digit numbers. Next we need to prove 9 is the greatest. Pick two numbers extremely close to each other, for example 987654321 and 987654312. Their difference is 9, and their common divisor can only be a factor of 9. Therefore, 9 is the greatest common divisor for these two numbers. As a result, 9 is the greatest common divisor for all these 362880 numbers.

13. Given the following vertical algorithm of multiplication between two numbers, if the blanks can only be filled with prime numbers, what is the two numbers that are being multiplied together? (Only write out the two numbers, not the product).



【Solution】Single digit prime numbers only include 2, 3, 5 and 7. The pair of unit digits has a product whose unit digit is still a prime number, therefore this pair can only be (3,5) or (5,7). From here, deducting in the order of unit digit – tens digit – hundreds digit, the answer is $775\times33$.

14. A 5-digit number gives different remainders when divided by 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. What is the largest possible value of this 5-digit number?

【Solution】Let this 5-digit number be A.
A divided by 1, the remainder is 0. (This applies to any whole number)
A divided by 2, the remainder is either 0 or 1. Since the remainders are required to be different, A divided by 2 gives remainder 1.
A divided by 3, the remainder is 0, 1 or 2. Since 0 and 1 are already taken, this third remainder must be 2.
Similarly, it can be found that divided by 1, 2, 3, …, 12, this number has remainder 0, 1, 2, …, 11.
Therefore, (A+1) is a multiple of 1, 2, 3, …, 12.
The least common multiple of these 12 numbers is:
$1\times2\times3\times4\times5\times7\times3\times11=27720$.

Since $27720\times3 < 99999 < 27720\times4$, the largest possible value of A is $27720\times3-1 = 83159$.

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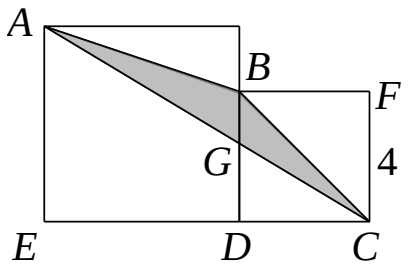
Questions 15 to 19 are worth 8 marks each.

15.Between the number 1000 to 1999 there are 1000 natural numbers, how many numbers are there in these 1000 numbers that have exactly two digits that are the same at the thousands, hundreds, tens and ones place?

【Solution】If the repeated digit is 1, one "1" is at the thousands place whereas the other "1" can be the hundreds, tens or unit digit. This gives $3\times9\times8=216$ numbers.
If the repeated digit is any number from 2 to 9, or 0, they will take two places among the hundreds, tens and ones place, the only place left has 8 choices, meaning there are $3\times8=24$ numbers for each of these 9 repeated digits.
Overall, there are $216+9\times24=432$ numbers satisfying the given condition.

16. The figure below is formed by a big square and a small square. If the side of the small square is 4cm, find the area of the shaded triangle *ABC* .




【Solution】Link AD. Triangle ABC and triangle DBC has equal height and the same base. As a result, they have the same area. Area DBC is $4\times4\div2=8$ , and therefore the answer is 8 cm².

17. The distance between Jurong port and Pasir Panjang port is 360km, it takes a ferry 35 hours to travel back and forth these two ports and it takes 5 hours more to travel against the current than along the current. Now, there is a motor boat that can travel 12km/h in still water. How long does it would take for the motor boat to travel back and forth the two ports?

【Solution】

The ferry takes $(35+5)\div 2=20$ hours to travel against the current, and $20-5=15$ hours to travel along the current. Therefore the ferry's speed against the current is $360\div 20=18$ km/h, and its speed along the current is $360\div 15=24$ km/h.

The speed of the current is $(24-18)\div 2=3$ km/h.

Hence, the motor boat's speed back and forth is (12-3) km/h and (12+3) km/h respectively. It would take the motor boat $360\div(12+3)+360\div(12-3)=64$ hours to travel back and forth the two ports.

18. Given that a, b, c are three distinct numbers in 1 to 9. If a, b, c can form six 3-digit numbers and the sum of five of them is 2234, what is the other 3-digit number?

【Solution】

The sum of all these six numbers is $222\times(a+b+c)$. Because $2234>222\times 10$, it is certain that $a+b+c>10$.

If $a+b+c=11$, then the remaining 3-digit number is $222\times 11-2234=208$, where $2+0+8=10\neq 11$, this is not the case.

If $a+b+c=12$, then the remaining 3-digit number is $222\times 12-2234=430$, where $4+3+0=7\neq 12$, this is not the case.

If $a+b+c=13$, then the remaining 3-digit number is $222\times 13-2234=652$, where $6+5+2=13$, this is a possible case.

If $a+b+c=14$, then the remaining 3-digit number is $222\times 14-2234=874$, where $8+7+4=19\neq 14$, this is not the case.

If $a+b+c\geq 15$, then the remaining 3-digit number is $\geq 222\times 15-2234=1096$, which is not possible.

In general, the only possible case is when $a+b+c=13$, the remaining 3-digit number is 652.

19. A natural number N can be written as the sum of 9 consecutive natural numbers, and can also be written as the sum of 10 consecutive natural numbers, and can also be written as the sum of 11 consecutive numbers, what is the smallest value of N?

【Solution】

The average of 9 consecutive natural numbers is at the center. N, as the sum of 9 consecutive natural numbers, must be a multiple of 9.

Similarly, the average of 11 consecutive natural numbers is at the center. N, as the sum of 11 consecutive natural numbers, must be a multiple of 11.

The average of 10 consecutive natural numbers is equal to 5 times the sum of the two numbers at the center. N, as the sum of 10 consecutive natural numbers, must be a multiple of 5.

The least common multiple of 9, 11 and 5 is $9\times 11\times 5=495$.

It can be verified that N=495 satisfies all the conditions.

Hence, 495 is the smallest value of N.

<u>Questions 20is worth 10 marks.</u>

20. In your opinion, from question 1 to 19, your favourite question is question _____ and the most difficult question is question _____.

(As long as your answer is within 1 to 19, you get full marks, otherwise you get zero.)